Property **UMVUE OR MVBUE**

UMVUE: Uniform Minimum Variance Unbiased Estimator

or

MVBUE:- Minimum Variance Bound Unbiased Estimator

It may be defined as if we have a

$$\frac{\partial \log l(x;\theta)}{\partial \theta} = A(\theta)\{\hat{\theta} - \tau(\theta)\}$$

Then $\hat{\theta}$ will be MVBUE of $\tau(\theta)$ and its variance given as:

$$V(\hat{\theta}) = \frac{(\tau'(\theta))^2}{A(\theta)}$$

**Procedure**

Step 1:- First we take the likelihood function of any p.d.f

Step 2:- Then taking the log of likelihood function of any p.d.f

Step 3:- Taking the differentiate log likelihood function with respect to parameter.

Step 4:- Finally we write as:

$$\frac{\partial \log l(x;\theta)}{\partial \theta} = A(\theta)\{\hat{\theta} - \tau(\theta)\}$$

$\frac{\partial \log l(x;\theta)}{\partial \theta}$ = A (function of parameter){estimator- $\tau$ (function of parameter)}

And

$$V(\hat{\theta}) = \frac{(\tau'(\theta))^2}{A(\theta)} \qquad \text{Or} \qquad V(\hat{\theta}) = \frac{\tau'(\theta)}{A(\theta)}$$

**Likelihood function**

The likelihood of ”n” random variable is defined to be the joint density of ”n” variables.

Let $x_1, x_2.........x_n$ be a random sample of size ”n” from a density $f(x;\theta)$ then L.H.S is def as

$$L(x;\theta) = f(x_1;\theta)f(x_2;\theta)f(x_3;\theta)....f(x_n;\theta) = \prod_{i=1}^{n} f(x_i;\theta)$$

Let “X” be a Possion distribution

$$f(x) = \frac{e^{-\theta}\theta^x}{x!}$$

Apply likelihood function

$$L(x,\theta) = \prod_{i=1}^{n}[\frac{e^{-\theta}\theta^x}{x!}] \qquad L(x,\theta) = \frac{e^{-n\theta}\theta^{\Sigma x_i}}{\prod_{i=1}^{n} x_i!}$$

Let X be a Gamma distribution $f(x) = \frac{x^{\alpha-1}}{\Gamma\alpha\beta^\alpha} e^{\frac{-x}{\beta}}$

Apply likelihood function $L(x,\alpha,\beta) = \frac{\prod_{i=1}^{n} x^{\alpha-1} e^{\frac{-\Sigma x}{\beta}}}{(\Gamma\alpha\beta^\alpha)^n}$

Probability function with likelihood and log likelihood functions

| f(x) | Parameter | Likelihood Function | Log Likelihood Function |
|---|---|---|---|
| $f(x)=e^{-\theta}\dfrac{\theta^{x}}{x!}$ | $\theta$ | $e^{-n\theta}\dfrac{\theta^{\Sigma x}}{\prod_{i=1}^{n} x!}$ | $-n\theta+\sum x\log\theta-\sum\log x!$ |
| $f(x)=\theta^{x}(1-\theta)^{1-x}$ | $\theta$ | $\theta^{\Sigma x}(1-\theta)^{n-\Sigma x}$ | $\sum x\log\theta+(n-\sum x)$ $\log(1-\theta)$ |
| $f(x)=\dfrac{1}{\sigma\sqrt{2\pi}}e^{\frac{1}{2}(\frac{x-\mu}{\sigma})^{2}}$ | $\mu,\sigma^{2}$ | $(\dfrac{1}{\sigma\sqrt{2\pi}})^{n}e^{-\frac{1}{2}\Sigma(\frac{x-\mu}{\sigma})^{2}}$ | $-n\log\sigma\sqrt{2\pi}-\dfrac{1}{2}\Sigma(\dfrac{x-\mu}{\sigma})^{2}$ |
| $f(x)=\dfrac{1}{\Gamma p\theta^{p}}x^{p-1}e^{-\frac{x}{\theta}}$ | $p;\theta$ | $(\dfrac{1}{\Gamma p\theta^{p}})^{n}(\prod_{i=1}^{n}x_i)^{p-1}e^{-\frac{\Sigma x}{\theta}}$ | $n\log\dfrac{1}{\Gamma p\theta^{p}}+(p-1)$ $\sum\log x-\dfrac{\sum x}{\theta}$ |

Q.1: Let $x_1, x_2, \ldots, x_n$ be a random sample of size ‘n’ from a normal distribution.i.e $x \approx N(\mu,\sigma^{2})$. Find MVBUE of μ and $\sigma^{2}$.also find its variance.

Q.2: Let “X” be Cauchy distribution then find its UMVUE if it exist.

Q.3: Let $X_1, X_2, \ldots, X_n$ be a random sample of size “n” from a normal distribution then. Find UMVUE for mean and variance also find their respective variances.

Q.4 From density function of $f(x)=\dfrac{1}{\Gamma P\theta^{P}}X^{P-1}e^{-X/\theta}\quad X\geq 0$

a) Show that MVBUE of $\theta$ is $\dfrac{\overline{X}}{n}$ with variance $\dfrac{\theta^{2}}{np}$ b) Show that MVBUE of $\dfrac{\partial\log\Gamma P}{\partial P}$ is $\dfrac{\sum\log x}{n}$ with variance $\dfrac{\frac{\partial^{2}\log\Gamma P}{\partial P^{2}}}{n}$.

Q.5: If “X” be Possion distribution then find the UMVBUE of $\theta$ also find its variance.

Q.6: Find MVUE of $\theta$ in binomial distribution also its variance.

Q.7. Find MVBUE of $\theta$ for the p.d.f $f(x)=\dfrac{1}{\theta}e^{x/\theta}$.

Q.8 Also find MVBUE of $\dfrac{1}{\theta}$ for the p.d.f of $f(x)=\theta e^{-x\theta}$ x>0.

Q.9: Find MVBUE of $\theta$ for $f(x)=\dfrac{1}{\sqrt{2\pi\sigma}}e^{-\frac{x^{2}}{2\sigma^{2}}}\quad -\infty<x<\infty$

Q.1: Let $x_1, x_2, \ldots, x_n$ be a random sample of size ‘n’ from a normal distribution.i.e $x \approx N(\mu,\sigma^{2})$. Find MVBUE of μ and $\sigma^{2}$.also find its variance.

**Solution**

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^{2}}$$

Applying likelihood function

$$L(\underline{x})=\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{n}e^{-\frac{1}{2\sigma^{2}}\Sigma(x-\mu)^{2}}$$

$$L(\underline{x}) = (\frac{1}{\sigma^2})^n (\frac{1}{\sqrt{2\pi}})^n e^{-\frac{1}{2\sigma^2}\Sigma(x-\mu)^2}$$

$$L(\underline{x}) = (\frac{1}{\sqrt{2\pi}})^n (\frac{1}{\sigma^2})^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\Sigma(x-\mu)^2}$$

Log likelihood function

$$\log L(\underline{x}) = n\log(\frac{1}{\sqrt{2\pi}}) + \frac{n}{2}\log(\frac{1}{\sigma^2}) - \frac{1}{2\sigma^2}\sum(x-\mu)^2$$

$$\log L(\underline{x}) = n\log(\frac{1}{\sqrt{2\pi}}) - \frac{n}{2}\log\sigma^2 - \frac{1}{2}(\sigma^2)^{-1}\sum(x-\mu)^2$$

For µ

$$\frac{\partial \log L(\underline{X})}{\partial\mu} = 0 - 0 - \frac{1}{2}(\sigma^2)^{-1}.2\sum(x-\mu)(-1)$$

$$\frac{\partial \log L(\underline{X})}{\partial\mu} = \frac{1}{\sigma^2}(\sum x - n\mu)$$

$$\frac{\partial \log L(\underline{X})}{\partial\mu} = \frac{n}{\sigma^2}\left[\frac{\sum x}{n} - \mu\right]$$

$$\frac{\partial \log L(\underline{X})}{\partial\mu} = \frac{n}{\sigma^2}\left[\bar{x} - \mu\right]$$

Where $A(\theta) = \frac{n}{\sigma^2}$ $\quad \hat{\theta} = \bar{x}$ $\quad \tau(\theta) = \mu$ $\quad \tau'(\theta) = 1$ $\quad \text{var}(\hat{\theta}) = \frac{1}{\frac{n}{\sigma^2}} = \frac{\sigma^2}{n}$

Hence $\bar{x}$ is MVBUE of $\mu$ with variance $\frac{\sigma^2}{n}$

For $\sigma^2$

$$\frac{\partial \log L(\underline{X})}{\partial\sigma^2} = 0 - \frac{n}{2\sigma^2}(1) - \frac{1}{2}(-1)(\sigma^2)^{-2}\sum(x-\mu)^2(1)$$

$$\frac{\partial \log L(\underline{X})}{\partial\sigma^2} = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}\Sigma(x-\mu)^2$$

$$\frac{\partial \log L(\underline{X})}{\partial\sigma^2} == \frac{n}{2\sigma^4}\left[\frac{\Sigma(x-\mu)^2}{n} - \sigma^2\right] \qquad \text{(A)}$$

$$\frac{\partial \log l(x;\theta)}{\partial\theta} = A(\theta)\{\hat{\theta} - \tau(\theta)\} \qquad \text{(B)}$$

Now comparing eq.(A) and (B) ,then we get

$A(\theta) = \frac{n}{2\sigma^4}$ $\quad \hat{\theta} = \frac{\Sigma(x-\mu)^2}{n}$ $\quad \tau(\theta) = \sigma^2$ $\quad \tau'(\theta) = 1$ $\quad Var(\hat{\theta}) = \frac{1}{\frac{n}{2\sigma^4}} = \frac{2\sigma^4}{n}$

Hence $\frac{\Sigma(x-\mu)^2}{n}$ is MVBUE of $\sigma^2$ with variance $\frac{2\sigma^4}{n}$.

Q. 2: Let $x_1, x_2, \ldots, x_n$ be a random sample of size 'n' from a Cauchy distribution with p.d.f

$f(x) = \frac{1}{\pi(1+(x-\theta)^2)}$ $\quad -\infty < x < +\infty$ Then find MVBUE of $\theta$ if it exists

Solution:

$$f(x)=\frac{1}{\pi(1+(x-\theta)^2)}$$

Applying likelihood function

$$L(x)=\left(\frac{1}{\pi}\right)^n\prod_{i=1}^{n}\left(\frac{1}{1+(x-\theta)^2}\right)$$

Log likelihood function

$$\log L(\underline{x})=-n\log\pi-\sum\log\left(1+(x-\theta)^2\right)$$

Differentiate w.r.t $\theta$

$$\frac{\partial\log L(\underline{x})}{\partial\theta}=-0-\sum\left(\frac{1}{1+(x-\theta)^2}\right)2(x-\theta)(-1)$$

$$=\frac{2\sum(x-\theta)}{1+(x-\theta)^2}\qquad=\frac{2(\sum x-n\theta)}{1+(x-\theta)^2}$$

$$=\frac{2n}{1+(x-\theta)^2}\left[\frac{\sum x}{n}-\theta\right]$$

$$=\frac{2n}{1+(x-\theta)^2}\left[\bar{x}-\theta\right]$$

It is not verify because Cauchy distribution does not exist this equation.

Q. 4: Show that in a sample from $f(x)=\frac{1}{\Gamma p\theta^p}X^{p-1}\theta^{\frac{-X}{\theta}}\quad 0<X<\infty$

a) Show that MVBUE of $\theta$ is $\frac{\overline{X}}{p}$ with variance $\frac{\theta^2}{np}$ b) Show that $\frac{1}{n}\sum\log X_i$ is MVBUE of $\frac{\partial\log\sqrt{p}}{\partial p}$ with variance $\frac{\frac{\partial^2\log\sqrt{p}}{\partial p^2}}{n}$ Assuming $\theta=1$

**Solution:**

As $f(x)=\frac{1}{\Gamma p\theta^p}X^{p-1}\theta^{\frac{-X}{\theta}}$

Applying likelihood function

$$L(\underline{X})=\left(\frac{1}{\Gamma p\theta^p}\right)^n\left(\prod_{i=1}^{n}X\right)^{P-1}\theta^{\frac{-\sum X}{\theta}}$$

$$L(\underline{X})=n\log\left(\frac{1}{\Gamma p\theta^p}\right)+(p-1)\sum\log X-\frac{\sum X}{\theta}$$

$$L(\underline{X})=-n\log\Gamma p-n\log\theta^p+(p-1)\sum\log X-\frac{\sum X}{\theta}$$

$$L(\underline{X})=-n\log\Gamma p-np\log\theta+(p-1)\sum\log X-\frac{\sum X}{\theta}\qquad\text{(A)}$$

Differentiate W.R.T $\theta$

$$\frac{\partial \log L(\underline{\underline{X}})}{\partial \theta} = 0 - \frac{np}{\theta} + 0 + \frac{\sum X}{\theta^2}$$

$$\frac{\partial \log L(\underline{\underline{X}})}{\partial \theta} = \frac{\sum X}{\theta^2} - \frac{np}{\theta}$$

$$\frac{\partial \log L(\underline{\underline{X}})}{\partial \theta} = \frac{np}{\theta^2}\left[\frac{\sum X}{np} - \theta\right]$$

$$\frac{\partial \log L(\underline{\underline{X}})}{\partial \theta} = \frac{np}{\theta^2}\left[\frac{\overline{X}}{p} - \theta\right] \qquad \text{(B)}$$

$$\frac{\partial \log l(x;\theta)}{\partial \theta} = A(\theta)\{\hat{\theta} - \tau(\theta)\} \qquad \text{(C)}$$

Comparing eq. (A) and (B), then we get

$$A(\theta) = \frac{np}{\theta^2} \qquad \hat{\theta} = \frac{\overline{X}}{p} \qquad \tau(\theta) = \theta \qquad \tau'(\theta) = 1 \qquad \operatorname{var}(\hat{\theta}) = \frac{1}{\frac{np}{\theta^2}} = \frac{\theta^2}{np}$$

Hence in eq (A) we assume $\theta = 1$

$$L(\underline{X}) = -n\log\overline{)p} + (p-1)\sum \log X - \sum X$$

Differentiate W.R.T p

$$\frac{\partial \log L(\underline{X})}{\partial p} = -n\frac{\partial \log\overline{)p}}{\partial p} + \sum \log X - 0$$

$$\frac{\partial \log L(\underline{X})}{\partial p} = \sum \log X - n\frac{\partial \log\overline{)p}}{\partial p}$$

$$\frac{\partial \log L(\underline{X})}{\partial p} = n\left[\frac{\sum \log X}{n} - \frac{\partial \log\overline{)p}}{\partial p}\right] \qquad \text{(A)}$$

$$\frac{\partial \log l(x;\theta)}{\partial \theta} = A(\theta)\{\hat{\theta} - \tau(\theta)\} \qquad \text{(B)}$$

Comparing eq. (A) and (B), then we get

$$A(\theta) = n \qquad \hat{\theta} = \frac{\sum \log X_i}{n} \quad \tau(\theta) = \frac{\partial \log\overline{)p}}{\partial p} \qquad \tau'(\theta) = \frac{\partial^2 \log\overline{)p}}{\partial p^2} \; Var(\hat{\theta}) = \frac{\frac{\partial^2 \log\overline{)p}}{\partial p^2}}{n}$$

Hence $\frac{1}{n}\sum \log X$ is MVBUE of $\frac{\partial \log\overline{)p}}{\partial p}$ with variance $Var(\hat{\theta}) = \frac{\frac{\partial^2 \log\overline{)p}}{\partial p^2}}{n}$

**Q.9:**

Find MVBUE of $\theta$ for $f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}} \quad -\infty<x<\infty$

Solution: AS X $\approx$ N $(0,\delta^2)$

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}$$

Apply likelihood function

$$L(\underline{x})=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}$$

$$L(\underline{x})=\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{\sum x^2}{2\sigma^2}}$$

$$L(\underline{x})=\left(\frac{1}{\sqrt{2\pi}}\right)^n\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{\sum x^2}{2\sigma^2}}$$

Log likelihood function

$$\log L(\underline{x})=n\log\left(\frac{1}{\sqrt{2\pi}}\right)^n-\frac{n}{2}\log\sigma^2-\frac{\sum x^2}{2\sigma^2}$$

Differentiate w.r.t $\sigma^2$

$$\frac{\partial\log L(\underline{x})}{\partial\sigma^2}=0-\frac{n}{2\sigma^2}+\frac{\sum x^2}{2\sigma^4}$$

$$\frac{\partial\log L(\underline{x})}{\partial\sigma^2}=\frac{n}{2\sigma^4}\left(\frac{\sum x^2}{n}-\sigma^2\right) \qquad \text{(A)}$$

$$\frac{\partial\log l(x;\theta)}{\partial\theta}=A(\theta)\{\hat{\theta}-\tau(\theta)\} \qquad \text{(B)}$$

Comparing eq. (A) and (B), then we get

$$A(\theta)=\frac{n}{2\sigma^4} \qquad \hat{\theta}=\frac{\sum x^2}{n} \qquad \tau(\theta)=\sigma^2 \qquad \tau'(\theta)=1 \quad Var(\hat{\theta})=\frac{2\sigma^4}{n}$$

Hence $\hat{\theta}=\frac{\sum x^2}{n}$ is MVBUE of $\sigma^2$ with variance $Var(\hat{\theta})=\frac{2\sigma^4}{n}$